Killing symmetries of generalized Minkowski spaces. 3- Space-time translations in four dimensions

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Abstract

In this paper, we continue the study of the Killing symmetries of a N-dimensional generalized Minkowski space, i.e. a space endowed with a (in general non-diagonal) metric tensor, whose coefficients do depend on a set of non-metrical coodinates. We discuss here the translations in such spaces, by confining ourselves (without loss of generality) to the four-dimensional case. In particular, the results obtained

are specialized to the case of a "deformed" Minkowski space $\widetilde{M_4}$ (i.e. a pseudoeuclidean space with metric coefficients depending on energy).

1 INTRODUCTION

This is the third of a series of papers aimed at studying the Killing symmetries of a N-dimensional generalized Minkowski space, i.e. a space endowed with a (in general non-diagonal) metric tensor, whose coefficients do depend on a set of non-metrical coordinates. In the previous papers [1] and [2], we discussed both the infinitesimal-algebraic and the finite-group structure of the space-time rotations in such a space.

An example of a generalized Minkowski space is provided by the deformed space-time $\widetilde{M_4}$ of *Deformed Special Relativity* (DSR). DSR is a generalization of the *Standard Special Relativity* (SR) based on a "deformation" of space-time, assumed to be endowed with a metric whose coefficients depend on the energy of the process considered [3]. Such a formalism applies in principle to *all* four interactions (electromagnetic, weak, strong and gravitational) — at least as far as their nonlocal behavior and nonpotential part is concerned — and provides a metric representation of them (at least for the process and in the energy range considered) ([4]-[7]). Moreover, it was shown that DSR is actually a five-dimensional scheme, in the sense that the deformed Minkowski space can be naturally embedded in a larger Riemannian manifold, with energy as fifth dimension [8].

In this paper, concluding the line of formal-mathematical research started in [13] (followed by [12], [1] and [2]), we shall end our investigation by discussing the space-time translations in generalized Minkowski spaces. For simplicity's sake, we shall restrict us (without loss of generality) to the four-dimensional case, by specializing our results to the deformed space-time $\widetilde{M_4}$ of DSR.

The organization of the paper is as follows. In Sect. 2 we briefly review the formalism of DSR and of the deformed Minkowski space $\widetilde{M_4}$. The results obtained in [1] and [2] concerning the maximal Killing group of generalized N-dimensional Minkowski spaces are summarized in Sect. 3. Then, Sect. 4 gives a general treatment of translation coordinate transformations in 4-d. generalized Minkowski spaces. Sect. 5, concerning the Abelian Lie group $Tr.(3,1)_{DEF.}$ of deformed space-time translations in DSR4, is divided in three Subsections: Subsect. 5.1 is about the 5-d. representation of infinitesimal contravariant generators of $Tr.(3,1)_{DEF.}$; in Subsect. 5.2 the "mixed" deformed Poincarè algebra is obtained, and 4-d. deformed Poincarè algebra fully explicited; finally Subsect. 5.3 explicits the form of infinitesimal and finite deformed translations in DSR4.

2 DEFORMED SPECIAL RELATIVITY IN FOUR DIMENSIONS (DSR4)

The generalized ("deformed") Minkowski space $\widetilde{M_4}$ (DMS4) is defined as a space with the same local coordinates x of M_4 (the four-vectors of the usual Minkowski space), but with metric given by the metric tensor[1]

[1] In the following, we shall employ the notation "ESC on" ("ESC off") to mean that the Einstein sum convention on repeated indices is (is not) used.

$$
\begin{aligned}
g_{\mu\nu,DSR4}(x^5) &= diag(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)) = \\
&\overset{\text{ESC off}}{=} \delta_{\mu\nu}\left[b_0^2(x^5)\delta_{\mu 0} - b_1^2(x^5)\delta_{\mu 1} - b_2^2(x^5)\delta_{\mu 2} - b_3^2(x^5)\delta_{\mu 3}\right],
\end{aligned}
\tag{1}
$$

where the $\left\{b_\mu^2(x^5)\right\}$ are dimensionless, real, positive functions of the independent, non-metrical (n.m.) variable x^5 [2]. The generalized interval in $\widetilde{M_4}$ is therefore given by ($x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, with c being the usual light speed in vacuum)

$$
\begin{aligned}
ds^2(x^5) &= b_0^2(x^5)c^2dt^2 - (b_1^2(x^5)dx^2 + b_2^2(x^5)dy^2 + b_3^2(x^5)dz^2) = \\
&= g_{\mu\nu,DSR4}(x^5)dx^\mu dx^\nu = dx * dx.
\end{aligned}
\tag{2}
$$

The last step in (2) defines the (x^5-dependent) scalar product $*$ in the deformed Minkowski space $\widetilde{M_4}$. In order to emphasize the dependence of DMS4 on the variable x^5, we shall sometimes use the notation $\widetilde{M_4}(x^5)$. It follows immediately that it can be regarded as a particular case of a Riemann space with null curvature.

From the general condition

$$
g_{\mu\nu,DSR4}(x^5)g^{\nu\rho}_{DSR4}(x^5) = \delta_\mu{}^\rho \tag{3}
$$

[2]Such a coordinate is to be interpreted as the energy (see Refs. [3]-[8]); moreover, the index 5 explicitly refers to the above-mentioned fact that the deformed Minkowski space can be *"naturally" embedded* in a five-dimensional (Riemannian) space [8].

we get for the contravariant components of the metric tensor

$$g^{\mu\nu}_{DSR4}(x^5) = diag(b_0^{-2}(x^5), -b_1^{-2}(x^5), -b_2^{-2}(x^5), -b_3^{-2}(x^5)) =$$

$$\stackrel{\text{ESC off}}{=} \delta^{\mu\nu}\left(b_0^{-2}(x^5)\delta^{\mu 0} - b_1^{-2}(x^5)\delta^{\mu 1} - b_2^{-2}(x^5)\delta^{\mu 2} - b_3^{-2}(x^5)\delta^{\mu 3}\right). \tag{4}$$

Let us stress that metric (1) is supposed to hold at a *local* (and not global) scale. We shall therefore refer to it as a "*topical*" deformed metric, because it is supposed to be valid not everywhere, but only in a suitable (local) space-time region (characteristic of both the system and the interaction considered).

The two basic postulates of DSR4 (which generalize those of standard SR) are [3]:

1- *Space-time properties:* Space-time is homogeneous, but space is not necessarily isotropic; a reference frame in which space-time is endowed with such properties is called a "*topical" reference frame* (TIRF). Two TIRF's are in general moving uniformly with respect to each other (i.e., as in SR, they are connected by a "inertiality" relation, which defines an equivalence class of ∞^3 TIRF);

2- *Generalized Principle of Relativity* (or *Principle of Metric Invariance*): All physical measurements within each TIRF must be carried out via the *same* metric.

The metric (1) is just a possible realization of the above postulates. We re-

fer the reader to Refs. [3]-[7] for the explicit expressions of the phenomenological energy-dependent metrics for the four fundamental interactions[3].

3 MAXIMAL KILLING GROUP OF GENERALIZED MINKOWSKI SPACES

A N-dimensional *generalized Minkowski space* $\widetilde{M_N}(\{x\}_{n.m.})$ is a Riemann space endowed with the global metric structure [1]

$$ds^2 = g_{\mu\nu}(\{x\}_{n.m.})dx^\mu dx^\nu \tag{5}$$

where the (in general non-diagonal) metric tensor $g_{\mu\nu}(\{x\}_{n.m.})$ $(\mu, \nu = 1, 2, ..., N)$ depends on a set $\{x\}_{n.m.}$ of $N_{n.m.}$ non-metrical coordinates (i.e. different from the N coordinates related to the dimensions of the space considered). We shall assume the (not necessarily hyperbolic) metric signature (T, S) (T timelike dimensions and $S = N - T$ spacelike dimensions). It follows that $\widetilde{M_N}(\{x\}_{n.m.})$ is *flat*, because all the components of the Riemann-Christoffel tensor vanish.

Of course, an example is just provided by the 4-d. deformed Minkowski space $\widetilde{M_4}(x^5)$ discussed in the previous Section.

The general form of the $N(N+1)/2$ Killing equations in a N-dimensional generalized Minkowski space is given by [1] (μ, $\nu = 1, 2, ..., N$, and $\mathbf{x}$ denotes the usual contravariant coordinate N-vector):

$$\xi_\mu(\mathbf{x})_{,\nu} + \xi_\nu(\mathbf{x})_{,\mu} = 0 \equiv \frac{\partial \xi_\mu(\mathbf{x})}{\partial x^\nu} + \frac{\partial \xi_\nu(\mathbf{x})}{\partial x^\mu} = 0. \tag{6}$$

Eqs. (6) are trivially satisfied by constant, covariant N-vectors ξ_μ — corresponding to the infinitesimal tranformation vectors of the space-time, N-parameter translation (Lie) group $Tr.(S, T = N - S)_{GEN.}$ of the generalized

[3]Since the metric coefficients $b^2_\mu(x^5)$ are *dimensionless*, they actually do depend on the ratio $\frac{x^5}{x^5_0}$, where

$$x^5_0 \equiv \ell_0 E_0$$

is a *fundamental length*, proportional (by the *dimensionally-transposing* constant ℓ_0) to the *threshold energy* E_0, characteristic of the interaction considered (see Refs. [3]-[8]).

Minkowskian space considered — entering the general expression of an infinitesimal translation [1]

$$tr.(S,T=N-S)_{GEN.} \ni \delta g:$$

$$x^{\mu} \rightarrow (x')^{\mu}(\mathbf{x},\{x\}_{n.m.}) = (x^{\mu})'(\mathbf{x},\{x\}_{n.m.}) =$$

$$= x^{\mu} + \delta x^{\mu}_{(g)}(\{x\}_{n.m.}) = x^{\mu} + \xi^{\mu}_{(g)}(\{x\}_{n.m.}), \tag{7}$$

where $tr.(S,T=N-S)_{GEN.}$ is the (Lie) algebra of N-d. space-time translations. Here, the contravariant N-vector $\delta x^{\mu}_{(g)}(\{x\}_{n.m.}) = \xi^{\mu}_{(g)}(\{x\}_{n.m.})$ is constant, i.e. independent of x^{μ}.

Therefore, the maximal Killing group of $\widetilde{M_N}(\{x\}_{n.m.})$ is the *generalized Poincaré* (or *inhomogeneous Lorentz*) *group* $P(S,T)^{N(N+1)/2}_{GEN.}$

$$P(T,S)^{N(N+1)/2}_{GEN.} = SO(T,S)^{N(N-1)/2}_{GEN.} \otimes_s Tr.(T,S)^{N}_{GEN.}, \tag{8}$$

i.e. the (semidirect [4]) product of the Lie group of N-dimensional generalized space-time rotations (or N-d. generalized, homogeneous Lorentz group $SO(T,S)^{N(N-1)/2}_{GEN.}$) with $N(N-1)/2$ parameters, and of the Lie group of generalized N-dimensional space-time translations $Tr.(T,S)^{N}_{GEN.}$ with N parameters (see Ref. [1]).

[4] As already pointed out in Ref. [2] and as shall be explicitely derived (in the hyperbolically-signed case $N=4, S=3, T=1$ of DSR4, without loss of generality) in Subsect. 5.2, in general we have that

$$\exists \text{ at least } 1 \ (\mu,\nu,\rho) \in \{1,...,N\}^3 : [I^{\mu\nu}_{GEN.}(\{x\}_{n.m.}), \Upsilon^{\rho}_{GEN.}(\{x\}_{n.m.})] \neq 0, \ \forall \{x\}_{n.m.},$$

where $I^{\mu\nu}_{GEN.}(\{x\}_{n.m.})$ are generalized rotation infinitesimal generators (i.e. generators of $SO(T,S)^{N(N-1)/2}_{GEN.}$) and $\Upsilon^{\rho}_{GEN.}(\{x\}_{n.m.})$ are generalized translation infinitesimal generators (i.e. generators of $Tr.(T,S)^{N}_{GEN.}$). Therefore, the correct group product to be considered is the semidirect one (see e.g. Ref. [10]).

4 TRANSLATIONS IN 4-d. GENERALIZED MINKOWSKI SPACES

In the case $N = 4$, one can write the components of the covariant Killing 4-vector of a generic 4-d. generalized Minkowski space $\widetilde{M_4}(\{x\}_{n.m.})$ as [1] (by omitting, for simplicity's sake, the dependence on the group element $g \in Tr.(S, T = 4 - S)_{GEN.} \subset P(S, T = 4 - S)_{GEN.}$)

$$\begin{cases} \xi_0(\{x\}_{m.}) = -\zeta^1 x^1 - \zeta^2 x^2 - \zeta^3 x^3 + T^0, \\ \xi_1(\{x\}_{m.}) = \zeta^1 x^0 + \theta^2 x^3 - \theta^3 x^2 - T^1, \\ \xi_2(\{x\}_{m.}) = \zeta^2 x^0 - \theta^1 x^3 + \theta^3 x^1 - T^2, \\ \xi_3(\{x\}_{m.}) = \zeta^3 x^0 + \theta^1 x^2 - \theta^2 x^1 - T^3. \end{cases} \tag{9}$$

Thus, independently of the explicit form of the metric tensor, *all 4-d. generalized Minkowski spaces admit the same covariant Killing vector*. In particular, with the hyperbolic signature $(+,-,-,-)$ (namely $S = 3$, $T = 1$) of SR and DSR4, it can be shown ([1],[2]) that $\zeta = (\zeta^1, \zeta^2, \zeta^3)$ is the (Euclidean) 3-vector of the dimensionless parameters (i.e. "generalized rapidities") of a generalized 3-d. "boosts" and $\theta = (\theta^1, \theta^2, \theta^3)$ is the (Euclidean) 3-vector of the dimensionless parameters (i.e. generalized angles) of a generalized 3-d. true rotations, whereas

$$T_\mu = (T^0, -T^1, -T^2, -T^3) \tag{10}$$

is the covariant 4-vector of the length-dimensioned parameters of a generalized 4-d. translation [5].

[5]Let us notice an important fact, peculiar to the translation component $Tr.(3,1)_{GEN.}$ of the 4-d. $(3,1)$ generalized Poincarè group $P(3,1)_{GEN.}$.

The elements of the 4-d. $(3,1)$ generalized rotation group $SO(3,1)_{GEN.}$ correspond, both at infinitesimal [1] and finite [2] level, to coordinate transformations homogeneous in their arguments, i.e. in the "length-dimensioned" coordinate basis $\{x^\mu\}_{\mu=0,1,2,3}$. It is then clear that the generalized parametric (Euclidean) 3-vectors $\underline{\theta}(g)$ and $\underline{\zeta}(g)$ must be dimensionless. In the cases $S = 3, T = 1$ of SR (corresponding to M_4) and of DSR4

The inhomogeneity of the (infinitesimal) translation transformation (7) obviously implies that it *cannot* be represented by a 4×4 matrix (at the infinitesimal, and then at the finite, level), i.e. no 4-d. representation of the infinitesimal generators of $Tr.(3,1)_{GEN.}$ exists. However, it is possible to get a matrix representation of the infinitesimal generators of the generalized translation group $Tr.(S,T=4-S)_{GEN.} \subset P(S,T=4-S)_{GEN.}$ by introducing a fifth auxiliary coordinate [9] $y=1$, devoid of any physical or metric meaning. This fictitious extra coordinate is introduced to the only aim of parametrizing the non-homogeneous part of the coordinate transformations

([1],[2]) (corresponding to $\widetilde{M_4}\left(x^5\right)$) $\underline{\theta}(g)$ and $\underline{\zeta}(g)$ have been identified with the generalized true rotation angle and generalized "boost rapidities" 3-vectors, respectively. By using the dimension-transposing constant velocity c, it has been possible to introduce a "velocity-dimensioned" "boost" parametric 3-vector $\underline{v}(g)$, that is a contravariant 3-vector in the 3-d. physical space embedded in the 4-d. Minkowski space being considered ($E_3 \subset M_4$ in SR, and $\widetilde{E_3}\left(x^5\right) \subset \widetilde{M_4}\left(x^5\right)$ in DSR4, respectively).

Analogously, because of the fact that the elements of the 4-d. $(3,1)$ generalized translation group $Tr.(3,1)_{GEN.}$ correspond, both at infinitesimal and finite level, to purely inhomogeneous coordinate transformations, it is clear that the generalized translation parametric 4-vector $T^{\mu}(g)$ must be "length-dimensioned" and have a *"context-dependent"* geometric nature. For example it is a "standard" contravariant 4-vector $T^{\mu}_{SR}(g)$ of M_4 in SR, and a "deformed" contravariant 4-vector $T^{\mu}_{DSR4}(g)$ of $\widetilde{M_4}\left(x^5\right)$ in DSR4.

That is why 3-d. Euclidean scalar products $\underline{\theta}(g)\cdot\underline{S_{DSR4}}(x^5)$ and $\underline{\zeta}(g)\cdot\underline{K_{DSR4}}(x^5)$, and 4-d. "deformed" scalar product (ESC on)

$$T_{\mu,(DSR4)}(g)\Upsilon^{\mu}_{DSR4}(x^5) = \Upsilon^{\mu}_{DSR4}(x^5)g_{\mu\nu,DSR4}(x^5)T^{\nu}_{DSR4}(g,x^5)$$

do appear in Eq. (53), which expresses the general form of the 5×5 matrix corresponding to a finite transformation of the 4-d. "deformed" (inhomogeneous Lorentz) Poincarè group $P(3,1)_{DEF.}$.

Notice that the notation "$(DSR4)$" in $T_{\mu,(DSR4)}$ has been used to mean that actually, as expressed by Eq. (10), $T_{\mu}(g)$ is independent of the (4-d.) metric context being considered.

of 4-d. $(3,1)$ generalized Poincaré group $P(3,1)_{GEN.}$[6].

Then, following the notation of page 150 of Ref. [9][7], one can consider[8] the following 5-d. (matrix) representation of the infinitesimal generators[9]

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[6]Let us stress that the coordinate $y=1$ has a merely parametrizing meaning, that is it has to span the *"transformative degree of freedom"* associated to the inhomogeneous component of the (maximal) Killing group $P(3,1)_{GEN.}$ of the 4-d. $(3,1)$ generalized Minkowski space $\widetilde{M_4}\left(\{x\}_{n.m.}\right)$ being considered. In other words, $y=1$ has to express the translation component of the Poincarè generalized coordinate transformations of $\widetilde{M_4}\left(\{x\}_{n.m.}\right)$.

The coordinate y has also a null total differential, because it is constant:

$$y=1 \Rightarrow dy=0,$$

whence it has not physical nor metric meaning.

Moreover, the trivial process of *"dimensional embedding"* (4-d.→5-d.) of the (matrix) representation of infinitesimal generators of $SO(3,1)_{GEN.}$ group (as expressed by Eq. (28)), does *not* change the infinitesimal-algebraic structure in any way; this is because of the fact that matrix rows and columns corresponding to the "auxiliary coordinate" y do *not* "mix" with the homogeneous components of the coordinate transformations being considered.

More generally, the introduction of y is necessary to give an explicit $(N+1)$-d. (matrix) representation of the infinitesimal generators of the generalized translation group $Tr.(S,T=N-S)_{GEN.}$, and then to calculate the (representation-independent) $N(S,T)$ generalized "mixed" Poincarè algebra, i.e. the commutator-exploited algebraic structure between the infinitesimal generators of $SO(S,T=N-S)_{GEN.}$ and the infinitesimal generators of $Tr.(S,T=N-S)_{GEN.}$ (for the case $N=4, S=3, T=1$ of DSR4, see Subsect. 5.2).

[7]The only difference with Ref. [9] (treating the SR case) is an *overall* minus sign. This is fully justifiable assuming that the parametric contravariant 4-vector ε^{μ}, used in Eq. (6-5.35) of page 150 in Ref. [1], is the *opposite* of T^{μ}_{SSR4}; that is, by omitting, for simplicity's sake, the dependence on $g \in Tr(3,1)_{STD.} \subset P(3,1)_{STD.}$:

$$\varepsilon^{\mu} \equiv -T^{\mu}_{SSR4} = \left(-T^0, -T^1, -T^2, -T^3\right).$$

[8]This choice could now seem a bit arbitrary, but it will prove to be justified and self-consistent from the following results, obtained, without loss of generality, in the case of DSR4 (see also Footnote 10).

[9]In the following, the upper-case Latin indices have range $\{0,1,2,3,6\}$, where the index 6 labels the auxiliary coordinate:

$$x^6 \equiv x_6 = y = 1$$

Moreover, independently of the contravariant or covariant nature of infinitesimal gener-

$\left\{(\Upsilon_\mu)^A{}_B\right\}_{\mu=0,1,2,3}$ of the group $Tr.(3,1)_{GEN.}$:

$$\Upsilon_0 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \Upsilon_1 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{11}$$

$$\Upsilon_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \Upsilon_3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

Namely, the only non-zero components of the above 5-d. representative matrices are

$$(\Upsilon_0)^0{}_6 = (\Upsilon_1)^1{}_6 = (\Upsilon_2)^2{}_6 = (\Upsilon_3)^3{}_6 = 1, \tag{13}$$

or, equivalently:

$$(\Upsilon_\mu)^A{}_B = \delta_\mu{}^A \delta_{6B}. \tag{14}$$

From Eqs. (11) and (12) one easily find the following properties of the above considered 5-d. representation of the *covariant* infinitesimal generators of $Tr.(3,1)_{GEN.}$ (here and in the following, $0_{5-d.}$ and $1_{5-d.}$ denote the 5×5 zero and unity matrix, respectively):

$$(\Upsilon_\mu)^n = 0_{5-d.}, \forall n \geq 2 \Rightarrow \exp(\Upsilon_\mu) = 1_{5-d.} + \Upsilon_\mu; \tag{15}$$

$$[\Upsilon_\mu, \Upsilon_\nu] = 0_{5-d.}, \quad \forall\, (\mu, \nu) \in \{0,1,2,3\}^2; \tag{16}$$

$$\Upsilon_\mu \neq \Upsilon_\mu \left(\{x\}_{m.}, \{x\}_{n.m.}\right). \tag{17}$$

ators, contravariant and covariant indices in their (matrix) representations conventionally stand for row and column indices, respectively.

In the following, we shall see that, in the DSR4 case, the properties (15) and (16) still hold for 5-d. matrix representation of the *contravariant* infinitesimal deformed translation generators. Moreover, as it is clear from Eqs. (11)-(14), the considered 5-d. representation of the *covariant* infintesimal generators of $Tr.(3,1)_{GEN.}$ are independent of the metric tensor (namely, they are the same irrespective of the 4-d. generalized Minkowski space $\widetilde{M_4}(\{x\}_{n.m.})$ considered). On the contrary, the *contravariant* generators *do depend* on the generalized metric, since[10]

$$\Upsilon^{\mu} \stackrel{\text{ESC on}}{=} g^{\mu\rho}(\{x\}_{n.m.})\Upsilon_{\rho} = \Upsilon^{\mu}(\{x\}_{n.m.}). \tag{18}$$

5 THE GROUP $Tr.(3,1)_{DEF.}$ OF 4-d. DEFORMED TRANSLATIONS IN $\widetilde{M_4}(x^5)$

5.1 The 5-d. representation of infinitesimal contravariant generators

Let us consider the case of $Tr.(3,1)_{DEF.}$, i.e. the deformed space-time translation group of the 4-d. deformed Minkowski space $\widetilde{M_4}(x^5)$ of DSR4, whose metric tensor is given by Eq.(3). Then, on account of Eqs. (11)-(12) and (18), the considered 5-d. matrix representation of the infinitesimal *contravariant*

[10]The assumed explicit 5-d. representations (11) and (12) are justifiable with the following reasoning. Eq. (9) (see [1]) expresses the independence of $T_\mu(g)$ on the (geo)metric context being considered; instead, its *contravariant* form will in general be "context-dependent", of course (see Footnote 5):

$$T^{\mu}(g,\{x\}_{n.m.}) \stackrel{\text{ESC on}}{=} g^{\mu\nu}(\{x\}_{n.m.})T_{\nu}(g).$$

Whence, because of the fact that in general a 4-d. "context-dependent" scalar product

$$T_{\mu}(g)\Upsilon^{\mu}(\{x\}_{n.m.}) = T_{\mu}(g)g^{\mu\nu}(\{x\}_{n.m.})\Upsilon_{\nu} = T^{\mu}(g,\{x\}_{n.m.})\Upsilon_{\mu}$$

appear in the explicit form of 5-d. matrix of infinitesimal generalized translation (e.g., in DSR4 case, $T_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5)$ of Eq. (48)), it is then clear that the set of the *covariant* infinitesimal generalized translation generators has to be *"context-independent"*, i.e. has to be the same irrespective of the 4-d. generalized Minkowski space $\widetilde{M_4}(\{x\}_{n.m.})$ considered.

deformed translation generators reads

$$\Upsilon^{0}_{DSR4}(x^5) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & b_0^{-2}(x^5) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{19}$$

$$\Upsilon^{1}_{DSR4}(x^5) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_1^{-2}(x^5) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{20}$$

$$\Upsilon^{2}_{DSR4}(x^5) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_2^{-2}(x^5) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{21}$$

$$\Upsilon^{3}_{DSR4}(x^5) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_3^{-2}(x^5) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

The only non-zero components are therefore given by

$$\begin{aligned} \left(\Upsilon^{0}_{DSR4}(x^5)\right)^{0}_{\;\;6} &= -\frac{b_0^{-2}(x^5)}{b_1^{-2}(x^5)}\left(\Upsilon^{1}_{DSR4}(x^5)\right)^{1}_{\;\;6} = \\ &= -\frac{b_0^{-2}(x^5)}{b_2^{-2}(x^5)}\left(\Upsilon^{2}_{DSR4}(x^5)\right)^{2}_{\;\;6} = -\frac{b_0^{-2}(x^5)}{b_3^{-2}(x^5)}\left(\Upsilon^{3}_{DSR4}(x^5)\right)^{3}_{\;\;6} = b_0^{-2}(x^5), \end{aligned} \tag{23}$$

or equivalently (ESC off) :

$$\left(\Upsilon^{\mu}_{DSR4}(x^5)\right)^{A}_{\ B} = \left(b_0^{-2}(x^5)\delta^{\mu 0} - b_1^{-2}(x^5)\delta^{\mu 1} - b_2^{-2}(x^5)\delta^{\mu 2} - b_3^{-2}(x^5)\delta^{\mu 3}\right)\delta_{\mu}^{\ A}\delta_{6B}. \tag{24}$$

From Eqs. (19)-(22) one immediately gets the following (*representation-independent*) properties of the *contravariant* deformed translation infinitesimal generators in $\widetilde{M_4}(x^5)$:

$$\left(\Upsilon^{\mu}_{DSR4}(x^5)\right)^{n} = 0_{5-d.}, \forall n \geq 2 \Rightarrow \exp(\Upsilon^{\mu}_{DSR4}(x^5)) = 1_{5-d.} + \Upsilon^{\mu}_{DSR4}(x^5); \tag{25}$$

$$[\Upsilon^{\mu}_{DSR4}(x^5), \Upsilon^{\nu}_{DSR4}(x^5)] = 0_{5-d.}, \forall \mu, \nu \in \{0,1,2,3\}\,; \tag{26}$$

$$\Upsilon^{\mu}_{DSR4} = \Upsilon^{\mu}_{DSR4}(x^5), \Upsilon^{\mu}_{DSR4} \neq \Upsilon^{\mu}_{DSR4}(\{x\}_{m.}). \tag{27}$$

It follows from Eq. (26) that ($tr.(3,1)_{DEF.}$) $Tr.(3,1)_{DEF.}$ is a proper (subalgebra) Abelian subgroup of the 4-d. deformed Poincaré (algebra) group ($su(2)_{DEF.} \times su(2)_{DEF.} \times_s tr.(3,1)_{DEF.}$) $P(3,1)_{DEF.}$, whose infinitesimal (*contravariant*) generators (by Eq. (27)) are *independent* of the metric variables of $\widetilde{M_4}(x^5)$, but do (*parametrically*) depend on the *non-metric* variable x^5.

5.2 The "mixed" deformed Poincarè algebra and deformed Poincarè algebra $(su(2)_{DEF.} \times su(2)_{DEF.}) \times_s tr.(3,1)_{DEF.}$

It is now possible to find the "mixed" algebraic structure of the 4-d. deformed Poincaré group $P(3,1)_{DEF.}$ of DSR4; this can be exploited by evaluating the commutators (which, as in the case of the 4-d. deformed Lorentz algebra [1] $su(2)_{DEF.} \times su(2)_{DEF.}$, will be *representation-independent*) among the infinitesimal generators of $Tr.(3,1)_{DEF.}$ and the infinitesimal generators of the deformed homogeneous Lorentz group $SO(3,1)_{DEF.}$. To this aim, one has to represent the infinitesimal generators of $SO(3,1)_{DEF.}$ as 5×5 matrices

in the auxiliary fictitious 5-d. space with $y = 1$ as extra dimension. It is easy to see that this amounts to the following trivial replacement:

$$I^{\alpha\beta}_{DSR4}(x^5) \rightarrow \begin{pmatrix} I^{\alpha\beta}_{DSR4}(x^5) & 0 \\ 0 & 0 \end{pmatrix} \quad \forall\,(\alpha,\beta) \in \{0,1,2,3\}^2\,, \tag{28}$$

where $I^{\alpha\beta}_{DSR4}(x^5)$ are the infinitesimal generators of the 4-d. deformed homogeneous Lorentz group $SO(3,1)_{DEF.}$ of DSR4 in the 4-d. matrix representation derived in Ref. [1]. We have therefore:

$$I^{10}_{DSR4}(x^5) = \begin{pmatrix} 0 & -b_0^{-2}(x^5) & 0 & 0 & 0 \\ -b_1^{-2}(x^5) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{29}$$

$$I^{20}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & -b_0^{-2}(x^5) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -b_2^{-2}(x^5) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{30}$$

$$I^{30}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & -b_0^{-2}(x^5) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -b_3^{-2}(x^5) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{31}$$

$$I^{12}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b_1^{-2}(x^5) & 0 & 0 \\ 0 & b_2^{-2}(x^5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{32}$$

$$I^{23}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2^{-2}(x^5) & 0 \\ 0 & 0 & b_3^{-2}(x^5) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{33}$$

$$I^{31}_{DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1^{-2}(x^5) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -b_3^{-2}(x^5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

From Eqs. (19)-(22) and (29)-(34) one gets the following form for the 4-d. "mixed" deformed Poincarè algebra ($\forall\,(i,j,k) \in \{1,2,3\}^3$) :

$$\begin{cases} [I^{i0}_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] = b_0^{-2}(x^5)\Upsilon^i_{DSR4}(x^5); \\ \\ [I^{i0}_{DSR4}(x^5), \Upsilon^j_{DSR4}(x^5)] \overset{\text{ESC }\underline{\text{off}}\text{ on } i}{=} \delta^{ij}(x^5) b_i^{-2}(x^5)\Upsilon^0_{DSR4}(x^5); \\ \\ [I^{ij}_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] = 0; \\ \\ [I^{ij}_{DSR4}(x^5), \Upsilon^k_{DSR4}(x^5)] \overset{\text{ESC off }\underline{\text{on}}\ i \text{ and } j}{=} \delta^{ik} b_i^{-2}(x^5)\Upsilon^j_{DSR4}(x^5) - \delta^{jk} b_j^{-2}(x^5)\Upsilon^i_{DSR4}(x^5), \end{cases} \tag{35}$$

or in compact form ($\forall\,(\mu,\nu,\rho) \in \{0,1,2,3\}^3$):

$$[I^{\mu\nu}_{DSR4}(x^5), \Upsilon^\rho_{DSR4}(x^5)] =$$

$$= g^{\nu\rho}_{DSR4}(x^5)\Upsilon^\mu_{DSR4}(x^5) - g^{\mu\rho}_{DSR4}(x^5)\Upsilon^\nu_{DSR4}(x^5) =$$

$$\overset{\text{ES}\underline{\text{C}}\text{ off}}{=} \delta^{\nu\rho}\left(b_0^{-2}(x^5)\delta^{\nu 0} - b_1^{-2}(x^5)\delta^{\nu 1} - b_2^{-2}(x^5)\delta^{\nu 2} - b_3^{-2}(x^5)\delta^{\nu 3}\right)\Upsilon^\mu_{DSR4}(x^5) +$$

$$-\delta^{\mu\rho}\left(b_0^{-2}(x^5)\delta^{\mu 0} - b_1^{-2}(x^5)\delta^{\mu 1} - b_2^{-2}(x^5)\delta^{\mu 2} - b_3^{-2}(x^5)\delta^{\mu 3}\right)\Upsilon^\nu_{DSR4}(x^5). \tag{36}$$

Whence, in general

$$\exists \text{ at least } 1\ \ (\mu,\nu,\rho) \in \{0,1,2,3\}^3 : [I^{\mu\nu}_{DSR4}(x^5), \Upsilon^\rho_{DSR4}(x^5)] \neq 0,\ \forall x^5 \in R_0^+. \tag{37}$$

Therefore, although ($su(2)_{DEF.} \times su(2)_{DEF.}$) $SO(3,1)_{DEF.}$ and ($tr.(3,1)_{DEF.}$) $Tr.(3,1)_{DEF.}$ are proper (subalgebras) subgroups - non-Abelian and Abelian, respectively - of the 4-d. deformed Poincaré (algebra) group [1], they determine it only by their semidirect product (see e.g. Ref. [10]).

Let us change the basis of infinitesimal generators of $SO(3,1)_{DEF.}$ to the "self-representative" one by defining the following deformed space-time infinitesimal generator Euclidean 3-vectors ([1],[2]) ($\forall i = 1, 2, 3$):

$$S^i_{DSR4}(x^5) \overset{\text{ESC on}}{\equiv} \frac{1}{2}\epsilon^i{}_{jk} I^{jk}_{DSR4}(x^5), \tag{38}$$

$$K^i_{DSR4}(x^5) \equiv I^{0i}_{DSR4}(x^5), \tag{39}$$

where ϵ_{ijk} is the (Euclidean) Levi-Civita 3-tensor with the convention $\epsilon_{123} \equiv 1$. In this basis, the "mixed" part of the 4-d. deformed Poincaré algebra can be written as (ESC off) :

$$[K^i_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] = -b_0^{-2}(x^5)\Upsilon^i_{DSR4}(x^5);$$

$$[K^i_{DSR4}(x^5), \Upsilon^j_{DSR4}(x^5)] \overset{ESCoff\ on\ i}{=} -\delta^{ij} b_i^{-2}(x^5)\Upsilon^0_{DSR4}(x^5);$$

$$\begin{aligned}[S^i_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] &= [\frac{1}{2}\epsilon^i{}_{jk} I^{jk}_{DSR4}(x^5), \Upsilon^0_{DSR4}](x^5) = \\ &= \frac{1}{2}\epsilon^i{}_{jk}[I^{jk}_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] = 0;\end{aligned}$$

$$
\begin{aligned}
[S^i_{DSR4}(x^5), \Upsilon^k_{DSR4}(x^5)] &= [\frac{1}{2}\epsilon^i{}_{jl} I^{jl}_{DSR4}(x^5), \Upsilon^k_{DSR4}(x^5)] = \\
&= \frac{1}{2}\epsilon^i{}_{jl}[I^{jl}_{DSR4}(x^5), \Upsilon^k_{DSR4}(x^5)] = \\
&= \frac{1}{2}\epsilon^i{}_{jl}\left(\delta^{jk} b_j^{-2}(x^5)\Upsilon^l_{DSR4}(x^5) - \delta^{lk} b_l^{-2}(x^5)\Upsilon^j_{DSR4}(x^5)\right) \overset{\text{ESC off on } k}{=} \\
&= \frac{1}{2}\left(\epsilon^{ik}{}_l b_k^{-2}(x^5)\Upsilon^l_{DSR4}(x^5) - \epsilon^i{}_j{}^k b_k^{-2}(x^5)\Upsilon^j_{DSR4}(x^5)\right) = \\
&\overset{\text{ESC off on } k}{=} \epsilon_{ikl} b_k^{-2}(x^5)\Upsilon^l_{DSR4}(x^5).
\end{aligned}
\tag{40}
$$

On account of the results obtained in Ref. [1] for the 4-d. deformed (homogeneous) Lorentz algebra $su(2)_{DEF.} \times su(2)_{DEF.}$, we can write the whole Lie algebra $(su(2)_{DEF.} \times su(2)_{DEF.}) \times_s tr.(3,1)_{DEF.}$ of the 4-d. deformed Poincaré group $P(3,1)_{DEF.}$ (i.e. the algebraic-infinitesimal structure of the

maximal Killing group of $\widetilde{M_4}(x^5)$) as

4-d. deformed space-time
rotation algebra $su(2)_{DEF.} \times su(2)_{DEF.}$:

$$
\begin{aligned}
&[I^{\alpha\beta}_{DSR4}(x^5), I^{\rho\sigma}_{DSR4}(x^5)] = \\
&= g^{\alpha\sigma}_{DSR4}(x^5)I^{\beta\rho}_{DSR4}(x^5) + g^{\beta\rho}_{DSR4}(x^5)I^{\alpha\sigma}_{DSR4}(x^5)+ \\
&-g^{\alpha\rho}_{DSR4}(x^5)I^{\beta\sigma}_{DSR4}(x^5) - g^{\beta\sigma}_{DSR4}(x^5)I^{\alpha\rho}_{DSR4}(x^5) = \\
&\stackrel{\text{ESC off}}{=} \delta^{\alpha\sigma}(b_0^{-2}(x^5)\delta^{\alpha 0} - b_1^{-2}(x^5)\delta^{\alpha 1}+ \\
&-b_2^{-2}(x^5)\delta^{\alpha 2} - b_3^{-2}(x^5)\delta^{\alpha 3})I^{\beta\rho}_{DSR4}(x^5)+ \\
&+\delta^{\beta\rho}(\delta^{\beta 0}b_0^{-2}(x^5) - \delta^{\beta 1}b_1^{-2}(x^5)+ \\
&-\delta^{\beta 2}b_2^{-2}(x^5) - \delta^{\beta 3}b_3^{-2}(x^5))I^{\alpha\sigma}_{DSR4}(x^5)+ \\
&-\delta^{\alpha\rho}(\delta^{\alpha 0}b_0^{-2}(x^5) - \delta^{\alpha 1}b_1^{-2}(x^5)+ \\
&-\delta^{\alpha 2}b_2^{-2}(x^5) - \delta^{\alpha 3}b_3^{-2}(x^5))I^{\beta\sigma}_{DSR4}(x^5)+ \\
&-\delta^{\beta\sigma}(\delta^{\beta 0}b_0^{-2}(x^5) - \delta^{\beta 1}b_1^{-2}(x^5)+ \\
&-\delta^{\beta 2}b_2^{-2}(x^5) - \delta^{\beta 3}b_3^{-2}(x^5))I^{\alpha\rho}_{DSR4}(x^5)
\end{aligned}
\tag{41}
$$

4-d. deformed space-time translation algebra $tr.(3,1)_{DEF.}$: $[\Upsilon^{\mu}_{DSR4}(x^5), \Upsilon^{\nu}_{DSR4}(x^5)] = 0$ (42)

4-d. *"mixed"* deformed space-time roto-translational algebra :

$$\begin{cases} [I^{\mu\nu}_{DSR4}(x^5), \Upsilon^{\rho}_{DSR4}(x^5)] = \\ \\ = g^{\nu\rho}_{DSR4}(x^5)\Upsilon^{\mu}_{DSR4} - g^{\mu\rho}_{DSR4}(x^5)\Upsilon^{\nu}_{DSR4} = \\ \\ \stackrel{\text{ESC off}}{=} \delta^{\nu\rho}(b_0^{-2}(x^5)\delta^{\nu 0} - b_1^{-2}(x^5)\delta^{\nu 1} + \\ \\ -b_2^{-2}(x^5)\delta^{\nu 2} - b_3^{-2}(x^5)\delta^{\nu 3})\Upsilon^{\mu}_{DSR4}(x^5) + \\ \\ -\delta^{\mu\rho}(b_0^{-2}(x^5)\delta^{\mu 0} - b_1^{-2}(x^5)\delta^{\mu 1} + \\ \\ -b_2^{-2}(x^5)\delta^{\mu 2} - b_3^{-2}(x^5)\delta^{\mu 3})\Upsilon^{\nu}_{DSR4}(x^5), \end{cases} \tag{43}$$

or, in the "self-representation" deformed infinitesimal generator basis

$$\mathbf{S}_{DSR4}(x^5) \equiv (I^{23}_{DSR4}(x^5), I^{31}_{DSR4}(x^5), I^{12}_{DSR4}(x^5))$$

and

$$\mathbf{K}_{DSR4}(x^5) \equiv (I^{01}_{DSR4}(x^5), I^{02}_{DSR4}(x^5), I^{03}_{DSR4}(x^5)) :$$

4-d. deformed space-time
rotation algebra
$su(2)_{DEF.} \times su(2)_{DEF.}$:

$$\left\{\begin{array}{l}
[S^i_{DSR4}(x^5), S^j_{DSR4}(x^5)] \overset{\text{ESC off on } i \text{ and } j}{=} \\
\\
= \left(\sum_{s=1}^{3}(1-\delta_{is})((1-\delta_{js})b_s^{-2}(x^5)\right)\epsilon_{ijk}S^k_{DSR4}(x^5) = \\
\\
= \epsilon_{ijk}b_k^{-2}(x^5)S^k_{DSR4}(x^5) \\
\\
\\
[K^i_{DSR4}(x^5), K^j_{DSR4}(x^5)] = \\
\\
= -b_0^{-2}(x^5)\epsilon_{ijk}S^k_{DSR4}(x^5) \\
\\
\\
[S^i_{DSR4}(x^5), K^j_{DSR4}(x^5)] = \\
\\
\overset{\text{ESC off on } j}{=} \epsilon_{ijl}K^l_{DSR4}(x^5)\left(\sum_{s=1}^{3}\delta_{js}b_s^{-2}(x^5)\right) = \\
\\
\overset{\text{ESC off on } j}{=} \epsilon_{ijl}b_j^{-2}(x^5)K^l_{DSR4}(x^5)
\end{array}\right. \tag{44}$$

4-d. deformed space-time translation algebra $tr.(3,1)_{DEF.}$:

$$[\Upsilon^\mu_{DSR4}(x^5), \Upsilon^\nu_{DSR4}(x^5)] = 0 \tag{45}$$

4-d. *"mixed"* deformed space-time
roto-translational algebra :

$$\begin{cases} [K^i_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] = -b_0^{-2}(x^5)\Upsilon^i_{DSR4}(x^5) \\ \\ [K^i_{DSR4}(x^5), \Upsilon^j_{DSR4}(x^5)] \overset{\text{ESC } \underline{\text{off}} \text{ on } i}{=} -\delta^{ij} b_i^{-2}(x^5)\Upsilon^0_{DSR4}(x^5) \\ \\ [S^i_{DSR4}(x^5), \Upsilon^0_{DSR4}(x^5)] = 0 \\ \\ [S^i_{DSR4}(x^5), \Upsilon^k_{DSR4}(x^5)] \overset{\text{ESC } \underline{\text{off}} \text{ on } k}{=} \epsilon_{ikl} b_k^{-2}(x^5)\Upsilon^l_{DSR4}(x^5). \end{cases} \tag{46}$$

5.3 Explicit form of infinitesimal and finite deformed translations in DSR4

The 5×5 matrix $\mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5)$, representing the infinitesimal (i.e. algebraic) element[11] $\delta g \in tr.(3,1)_{DEF.} \subset ((su(2)_{DEF.} \times su(2)_{DEF.}) \times_s tr.(3,1)_{DEF.})$ which corresponds to a deformed, infinitesimal 4-d. space-time translation by a parametric, length-dimensioned (infinitesimal[12]) *contravariant* 4-vector

$$T^\mu_{DSR4}(g,x^5) \equiv g^{\mu\rho}_{DSR4}(x^5)T_\rho(g) = \left(b_0^{-2}(x^5)T^0, b_1^{-2}(x^5)T^1, b_2^{-2}(x^5)T^2, b_3^{-2}(x^5)T^3\right) \tag{47}$$

[11]For precision's sake, at the infinitesimal transformation level $\delta g \in tr.(3,1)_{DEF.} \subset ((su(2)_{DEF.} \times su(2)_{DEF.}) \times_s tr.(3,1)_{DEF.})$ should be substituted for $g \in Tr.(3,1)_{DEF.} \subset P(3,1)_{DEF.}$. But, for simplicity's sake, we will omit, but understand, this cumbersome notation.

[12]Needless to say, at the algebraic and group level, "length-dimensioned" translation parameter contravariant deformed 4-vectors $T^\mu_{DSR4}(g,x^5)$ will be infinitesimal and finite, respectively. This will be understood, and, for simplicity's sake, no notational distinction will be made.

As will be explicitly seen later (in the DSR4 case, without loss of generality), the infinitesimal or finite nature of translation parameter N-vectors (such as $T^\mu_{DSR4}(g,x^5)$ in DSR4) is in general the only difference between algebraic and group level in translation coordinate transformations in a generalized N-d. Minkowski space $\widetilde{M_N}\left(\{x\}_{n.m.}\right)$.

in $\widetilde{M_4}(x^5)$, is defined by[13]

$$\begin{aligned}
\mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5) &\equiv T^{\mu}_{DSR4}(g,x^5)\Upsilon_{\mu,(DSR4)} = \\
&= g^{\mu\rho}_{DSR4}(x^5)T_{\rho}(g)\Upsilon_{\mu,(DSR4)} = T_{\mu}(g)\Upsilon^{\mu}_{DSR4}(x^5) = \\
&= T^0(g)\Upsilon^0_{DSR4}(x^5) - T^1(g)\Upsilon^1_{DSR4}(x^5) + \\
&\quad -T^2(g)\Upsilon^2_{DSR4}(x^5) - T^3(g)\Upsilon^3_{DSR4}(x^5) \qquad (48)
\end{aligned}$$

or, explicitly:

$$\mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5) = \begin{pmatrix} 0 & 0 & 0 & 0 & b_0^{-2}(x^5)T^0(g) \\ 0 & 0 & 0 & 0 & b_1^{-2}(x^5)T^1(g) \\ 0 & 0 & 0 & 0 & b_2^{-2}(x^5)T^2(g) \\ 0 & 0 & 0 & 0 & b_3^{-2}(x^5)T^3(g) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (49)$$

Therefore, the 4-d. deformed, infinitesimal space-time translation by (infinitesimal)

$$T^{\mu}_{DSR4}(g,x^5) \overset{\text{ESC on}}{\equiv} g^{\mu\rho}_{DSR4}(x^5)T_{\rho}(g)$$

in $\widetilde{M_4}(x^5)$ - corresponding to

$$\delta g \in tr.(3,1)_{DEF.} \subset ((su(2)_{DEF.} \times su(2)_{DEF.}) \times_s tr.(3,1)_{DEF.}) -$$

[13] The parentheses in the notation "$DSR4$" in $\Upsilon_{\mu,(DSR4)}$ denote that, as expressed by Eqs. (11)-(12) and above discussed - see also Footnote 10 -, the deformed translation infinitesimal *covariant* generators are independent of the (geo)metric context being considered.

is given by (ESC on; for simplicity's sake, dependence on $\{x_{m.}\}$ is omitted):

$$\begin{pmatrix} (x')^0_{(g),DSR4}(x^0,x^5) \\ (x')^1_{(g),DSR4}(x^1,x^5) \\ (x')^2_{(g),DSR4}(x^2,x^5) \\ (x')^3_{(g),DSR4}(x^3,x^5) \\ y'_{DSR4} \end{pmatrix} = \begin{pmatrix} (x^0)'_{(g),DSR4}(x^0,x^5) \\ (x^1)'_{(g),DSR4}(x^1,x^5) \\ (x^2)'_{(g),DSR4}(x^2,x^5) \\ (x^3)'_{(g),DSR4}(x^3,x^5) \\ y'_{DSR4} \end{pmatrix} =$$

$$\overset{\text{ESC on}}{=} \left(1_{5-d.} + \mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5)\right)^A{}_B \, x^B =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & b_0^{-2}(x^5)T^0(g) \\ 0 & 1 & 0 & 0 & b_1^{-2}(x^5)T^1(g) \\ 0 & 0 & 1 & 0 & b_2^{-2}(x^5)T^2(g) \\ 0 & 0 & 0 & 1 & b_3^{-2}(x^5)T^3(g) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ (y=)\,1 \end{pmatrix} =$$

$$= \begin{pmatrix} x^0 + b_0^{-2}(x^5)T^0(g) \\ x^1 + b_1^{-2}(x^5)T^1(g) \\ x^2 + b_2^{-2}(x^5)T^2(g) \\ x^3 + b_3^{-2}(x^5)T^3(g) \\ (y=)\,1 \end{pmatrix}. \tag{50}$$

At the finite transformation level, one has to evaluate the exponential of the matrix $\mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5)$, i.e. the 5×5 matrix $\exp\left(\mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5)\right)$, representing the finite (i.e. group) element $g \in Tr.(3,1)_{DEF.} \subset SO(3,1)_{DEF.}$ which corresponds to a deformed, finite 4-d. space-time translation by a parametric, length-dimensioned (finite) *contravariant* 4-vector $T^\mu_{DSR4}(g,x^5) \overset{\text{ESC on}}{\equiv} g^{\mu\rho}_{DSR4}(x^5)T_\rho(g)$:

$$\begin{aligned} \exp\left(\mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5)\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5)\right)^n = \\ &= 1_{5-d.} + \mathcal{T}_{T^\mu_{DSR4}(g,x^5),DSR4}(x^5). \end{aligned} \tag{51}$$

Then, as anticipated in Footnote 12, for translation transformations the only difference between the infinitesimal level and the finite one is provided by the infinitesimal or finite nature of the *contravariant* translation parameters $T^{\mu}_{DSR4}(g, x^5)$. Such a result is a peculiar feature of the space-time translation component of the 4-d. deformed Poincaré group $P(3,1)_{DEF.}$, and in general of translation coordinate transformations in N-d. generalized Minkowski spaces $\widetilde{M_N}\left(\{x\}_{n.m.}\right)$. Still considering the case of DSR4, it can be recovered also by exploiting the Abelian nature of $Tr.(3,1)_{DEF.}$ and the property of the powers of its infinitesimal generators (see Eq. (25)), and by using the

Baker-Campbell-Hausdorff formula [11]; one has indeed:

$$\exp\left(\mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5)\right) =$$

$$= \exp(T^0(g)\Upsilon^0_{DSR4}(x^5) - T^1(g)\Upsilon^1_{DSR4}(x^5) +$$

$$-T^2(g)\Upsilon^2_{DSR4}(x^5) - T^3(g)\Upsilon^3_{DSR4}(x^5)) =$$

$$= \exp\left(T^0(g)\Upsilon^0_{DSR4}(x^5)\right) \times$$

$$\times \exp\left(T^1(g)\Upsilon^1_{DSR4}(x^5)\right) \times \exp\left(T^2(g)\Upsilon^2_{DSR4}(x^5)\right) \times \exp\left(T^3(g)\Upsilon^3_{DSR4}(x^5)\right) =$$

$$= \left(\sum_{n=0}^{\infty} \frac{(T^0(g))^n}{n!}\left(\Upsilon^0_{DSR4}(x^5)\right)^n\right) \times \left(\sum_{n=0}^{\infty} \frac{(T^1(g))^n}{n!}\left(\Upsilon^1_{DSR4}(x^5)\right)^n\right) \times$$

$$\times \left(\sum_{n=0}^{\infty} \frac{(T^2(g))^n}{n!}\left(\Upsilon^2_{DSR4}(x^5)\right)^n\right) \times \left(\sum_{n=0}^{\infty} \frac{(T^3(g))^n}{n!}\left(\Upsilon^3_{DSR4}(x^5)\right)^n\right) =$$

$$= \left(1_{5-d.} + T^0(g)\Upsilon^0_{DSR4}(x^5)\right) \times \left(1_{5-d.} + T^1(g)\Upsilon^1_{DSR4}(x^5)\right) \times$$

$$\times \left(1_{5-d.} + T^2(g)\Upsilon^2_{DSR4}(x^5)\right) \times \left(1_{5-d.} + T^3(g)\Upsilon^3_{DSR4}(x^5)\right) =$$

$$= 1_{5-d.} + T^0(g)\Upsilon^0_{DSR4}(x^5) + T^1(g)\Upsilon^1_{DSR4}(x^5) + T^2(g)\Upsilon^2_{DSR4}(x^5) +$$

$$+T^3(g)\Upsilon^3_{DSR4}(x^5) =$$

$$1_{5-d.} + \mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5). \tag{52}$$

On account of the non-commutativity of the infinitesimal generators of $Tr.(3,1)_{DEF.}$ and of $SO(3,1)_{DEF.}$ (see Eqs. (36) and (37)), by using the

Baker-Campbell-Hausdorff formula [11], it is possible to state the following unequality for the considered 5-d. matrix representing the finite (i.e. group) element $g \in P(3,1)_{DEF.}$ corresponding to a finite, 4-d. deformed space-time roto-translation in $\widetilde{M_4}(x^5)$, of dimensionless parametric deformed angular (Euclidean) 3-vector $\theta(g)$, dimensionless parametric deformed "rapidity" (Euclidean) 3-vector $\zeta(g)$ and "length-dimensioned" parametric translational contravariant (deformed) 4-vector $T^{\mu}_{DSR4}(g,x^5) \equiv g^{\mu\rho}_{DSR4}(x^5)T_{\rho}(g)$ [14]:

$$
\begin{gathered}
\exp\left(-\theta(g)\cdot\mathbf{S}_{DSR4}(x^5) - \zeta(g)\cdot\mathbf{K}_{DSR4}(x^5) + T_{\mu}(g)\Upsilon^{\mu}_{DSR4}(x^5)\right) \neq \\
\neq \exp\left(-\theta(g)\cdot\mathbf{S}_{DSR4}(x^5) - \zeta(g)\cdot\mathbf{K}_{DSR4}(x^5)\right) \times \exp\left(T_{\mu}(g)\Upsilon^{\mu}_{DSR4}(x^5)\right) = \\
= \exp\left(-\theta(g)\cdot\mathbf{S}_{DSR4}(x^5) - \zeta(g)\cdot\mathbf{K}_{DSR4}(x^5)\right) \times \exp\left(\mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5)\right) = \\
= \exp\left(-\theta(g)\cdot\mathbf{S}_{DSR4}(x^5) - \zeta(g)\cdot\mathbf{K}_{DSR4}(x^5)\right) \times \left(1_{5-d.} + \mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5)\right) \neq \\
\neq \exp\left(-\theta(g)\cdot\mathbf{S}_{DSR4}(x^5)\right) \times \exp\left(-\zeta(g)\cdot\mathbf{K}_{DSR4}(x^5)\right) \times \left(1_{5-d.} + \mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5)\right) \neq \\
\neq \exp\left(-\theta^1(g)S^1_{DSR4}(x^5)\right) \times \exp\left(-\theta^2(g)S^2_{DSR4}(x^5)\right) \times \\
\times \exp\left(-\theta^3(g)S^3_{DSR4}(x^5)\right) \times \exp\left(-\zeta^1(g)K^1_{DSR4}(x^5)\right) \times \\
\times \exp\left(-\zeta^2(g)K^2_{DSR4}(x^5)\right) \times \exp\left(-\zeta^3(g)K^3_{DSR4}(x^5)\right) \times \left(1_{5-d.} + \mathcal{T}_{T^{\mu}_{DSR4}(g,x^5),DSR4}(x^5)\right),
\end{gathered}
\tag{53}
$$

where in the last two lines the non commutativity of deformed true rotation and "boost" infinitesimal generators have been used (see Eqs. (41) and (44)).

[14] Let us instead notice that, at infinitesimal level, all transformations of the Lie group $P(3,1)_{DEF.}$ commute.

By comparing Eq. (50) with the expression of a translation in the usual Minkowski space M_4 of SR (as before ESC on and, for simplicity's sake, dependence on $\{x_{m.}\}$ is omitted)

$$\begin{pmatrix} (x')^{0}_{(g),SSR4}\left(x^0\right) \\ (x')^{1}_{(g),SSR4}\left(x^1\right) \\ (x')^{2}_{(g),SSR4}\left(x^2\right) \\ (x')^{3}_{(g),SSR4}\left(x^3\right) \\ y'_{SSR4} \end{pmatrix} = \begin{pmatrix} \left(x^0\right)'_{(g),SSR4}\left(x^0\right) \\ \left(x^1\right)'_{(g),SSR4}\left(x^1\right) \\ \left(x^2\right)'_{(g),SSR4}\left(x^2\right) \\ \left(x^3\right)'_{(g),SSR4}\left(x^3\right) \\ y'_{SSR4} \end{pmatrix} =$$

$$= \left(1_{5-d.} + \mathcal{T}_{T^{\mu}_{SSR4}(g),SSR4}\right)^{A}{}_{B}\, x^B =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & T^0(g) \\ 0 & 1 & 0 & 0 & T^1(g) \\ 0 & 0 & 1 & 0 & T^3(g) \\ 0 & 0 & 0 & 1 & T^3(g) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ (y =)\, 1 \end{pmatrix} = \begin{pmatrix} x^0 + T^0(g) \\ x^1 + T^1(g) \\ x^2 + T^2(g) \\ x^3 + T^3(g) \\ (y =)\, 1 \end{pmatrix}, \qquad (54)$$

it is easily seen that passing from SR to DSR4 - i.e. *locally deforming and spatially anisotropizing* M_4 - (as far as space-time translations are concerned) amounts to the following parameter change:

$$\begin{aligned} T^{\mu}_{SSR4}(g) &= \left(T^0(g), T^1(g), T^2(g), T^3(g)\right) \rightarrow T^{\mu}_{DSR4}(g, x^5) = \\ &= \left(b_0^{-2}(x^5)T^0(g), b_0^{-2}(x^5)T^1(g), b_0^{-2}(x^5)T^2(g), b_0^{-2}(x^5)T^3(g)\right) \equiv \widetilde{T}^{\mu}_{DSR4}(g, x^5). \end{aligned} \qquad (55)$$

Then, extending the meaning of "*effective*" transformation parameters [2] to translation ones, we can say that in the translational case the "length-dimensioned" deformed translation parameter (deformed) *contravariant* 4-vector $T^{\mu}_{DSR4}(g, x^5)$ coincides with the *effective* "length-dimensioned" de-

formed translation parameter (deformed) *contravariant* 4-vector[15] $\widetilde{T}^{\mu}_{DSR4}(g, x^5)$:

$$T^{\mu}_{DSR4}(g, x^5) = \widetilde{T}^{\mu}_{DSR4}(g, x^5). \tag{56}$$

This is peculiar feature of the translation component $Tr.(3,1)_{DEF.}$ of the 4-d. deformed Poincarè group $P(3,1)_{DEF.}$, and it is due to the following fact: while for 4-d. space-time rotations (homogeneous transformations in the coordinates) the deformed transformation parameter 3-vectors $\theta\,(g)$ and $\zeta\,(g)$ are Euclidean (see Refs. [1] and [2]), in the case of 4-d. deformed translations (inhomogeneous transformations in the coordinates) the deformed translation parameter *contravariant* 4-vector $T^{\mu}_{DSR4}(g, x^5)$ is "length-dimensioned" and deformed, i.e. *dependent on the metric structure being considered* (see also Footnotes 5, 10 and 13).

Let us finally stress that the *(local) "deformating anisotropizing"* generalization of SR corresponding to DSR4 is fully self-consistent at space-time translation level, too. As noticed also in Refs. [1], [2] and [12], it is easy to see that *all* the results obtained in the present work for the DSR4 level (i.e. about the Lie group $Tr.(3,1)_{DEF.}$ of space-time deformed translations in the 4-d. "deformed" Minkowski space $\widetilde{M_4}(x^5)$) reduce, in the limit DSR4→SR, i.e. in the limit

$$\begin{aligned} g_{\mu\nu,DSR4}(x^5) &= diag\left(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)\right) \rightarrow \\ &\rightarrow g_{\mu\nu,SR} = diag\,(1,-1,-1,-1) \Leftrightarrow \\ &\Leftrightarrow b_{\mu}^2(x^5) \rightarrow 1, \forall \mu = 0,1,2,3, \end{aligned} \tag{57}$$

to the well-known results of SR (i.e. about the Lie group $Tr.(3,1)_{STD.}$ of space-time translations in the 4-d. Minkowski space M_4 of Einstein's Special Relativity).

[15] The contravariant 4-vector identity (56) is due to the very fact that the passage SR→DSR4:

$$g_{\mu\nu,SR} = diag\,(1,-1,-1,-1) \rightarrow g_{\mu\nu,DSR4}(x^5) = diag\left(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5)\right)$$

preserves the diagonality of the 2-rank, symmetric metric 4-tensor, still *destroying* its isochrony and spatial isotropy.

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